

ATTORNEY DOCKET NO. 12406US01

IN THE UNITED STATES PATENT AND TRADEMARK OFFICE

TITLE OF THE INVENTION: RISK-ADJUSTED METHOD FOR PRICING
FINANCIAL DERIVATIVES

INVENTOR: MILAN KRATKA

ATTORNEYS: McANDREWS, HELD & MALLOY, LTD.
500 West Madison Street
34th Floor
Chicago, Illinois 60661
(312) 707-8889

657070-707070

Background Of The Invention

This invention relates to methods and apparatus for pricing financial derivatives.

Current pricing models for financial derivatives are based on two pricing methods: (1) the "Classic Arbitrage Argument" (CAA), or (2) "Monte-Carlo" method. The CAA states the return on a perfectly-hedged investment portfolio is precisely equal to the return on U.S. Treasury bonds. Under the CAA method, it is assumed that investors are capable of continuously hedging their portfolios in infinitesimal increments with no costs. Moreover, the CAA method assumes that the parameters of the model do not change during the lifetime of the portfolio and that the markets will always precisely follow the pricing model.

Pricing models based on the CAA method are often referred to as "Risk Neutral Pricing" (RNP) models, because no risks are assumed by the CAA method. Under the CAA method, management of all risks and all discrepancies between the pricing model and the real market is passed to the human users of the model. RNP models are implemented for production in a form of numerical solutions to partial differential equations via finite difference schemes, finite element methods, closed form formulas, or by various tree type methods, including binomial and trinomial ones. For example, the Black-Scholes pricing method is an example of a method that is based on the CAA method and idealistic RNP assumptions.

Monte Carlo pricing models, on the other hand, are based on statistical and probabilistic evaluation of possible future price scenarios. These scenarios are typically generated with the aid of random number generators. Evaluation of derivative prices is modeled via the CAA, discounted payoff method, or a combination of both. In either case, the value of the risk does not explicitly enter the evaluation formula.

Therefore, RNP pricing models do not include market risks and costs in the valuation of a financial derivative. As a result, these pricing models generate valuations of financial derivatives that are not completely consistent with actual market behavior. Although the risks have been reflected in the market prices by various non-mathematical methods, the lack of

appropriate mathematical modeling has precluded the explicit inclusion of risks into these pricing and hedging models.

Summary of Invention

The present invention overcomes the limitation of the prior art by introducing the risk
5 premiums into a pricing model for calculating values of derivative securities.

It is an object of the present invention to provide a pricing model for calculating values of
derivative securities that consider various risk premiums.

It is a further object of the present invention to provide a pricing model for calculating
values of derivative securities that is consistent with market behavior.

One or more of the foregoing objects are met in whole or in part by the present risk-
adjusted pricing method and computer apparatus. The method includes determining the
underlying security and derivative, determining the risks and trading costs associated with the
derivative, formulating the RAP equation for the derivative, solving the RAP equation for the
derivative and outputting a value of the derivative based on the solution of the RAP equation.
15 The computer apparatus employs this method in a financial analysis application. The system
includes an input unit for taking inputs of derivative characteristics and model parameters, a
processing unit for computing a value of the derivative based on the input characteristics and
parameters using a risk-adjusted pricing equation, and an output unit for displaying the value of
the derivative.

Description of the Drawings

Figure 1 is a flowchart of a preferred embodiment of the RAP evaluation software.

5 Figure 2 is an output generated by use of a preferred embodiment of the RAP evaluation software that describes the Black-Scholes implied volatility.

Figure 3 is an output generated by use of a preferred embodiment of the RAP evaluation software that describes the difference between the Black-Scholes call price and the risk-adjusted call price.

10 Figure 4 is an output generated by use of a preferred embodiment of the RAP evaluation software that describes the difference between the Black-Scholes delta and the risk-adjusted delta.

Figure 5 is an output generated by use of a preferred embodiment of the RAP evaluation software that describes the difference between the Black-Scholes gamma and the risk-adjusted gamma.

15 Figure 6 is the source code of a preferred embodiment of the RAP evaluation software written in C/C++ language.

Figure 7 the article by Milan Kratka, RISK, April 1998, referenced in the specification.

Description of the Preferred Embodiment

The present invention overcomes the limitation of the prior art by introducing the risk premiums into a pricing model for calculating values of derivative securities. A derivative security is a security that derives its value from another underlying security, commodity, or object. Derivative securities include futures, forwards, options on stock, options on index, options on futures, basket options, swaps, swaptions, and others.

Contrary to the current RNP models based on the CAA or Monte-Carlo, the present invention is based on the Advanced Arbitrage Argument (AAA), which incorporates the value of the various risks associated with derivatives into the valuation of those derivatives. The AAA states that the expected return on an investment should not only equal the return on the U.S. government bonds (as assumed by CAA), but should also include expected investment costs (such as ask bid spreads, brokerage fees, borrowing fees, etc.) and premiums for taking investment risks (such as implied volatility risk, positive portfolio variance risk, price gap risk, etc.). The AAA can be summarized by the following template pricing equation, which is valid at any step of time during the life of an investment:

(1)

$$\begin{array}{ccccccc} \text{Expected Return} & & & \text{Return} & & \text{Rate of Expected} & \text{Rate of Expected} \\ \text{on} & = & & \text{on} & + & \text{Investment} & + \\ \text{Investment} & & & \text{Government Bonds} & & \text{Costs} & \text{Risk} \\ & & & & & & \text{Premium} \end{array}$$

Return on government bonds is often referred to as the “risk-free rate.” Expected risk premium compensates the investor for taking an exposure to various risks associated with the investment, similar in function and form to insurance premiums. The present invention implements the AAA through Risk Adjusted Pricing (RAP) methodology, which provides a means to quantitatively enumerate individual risk premiums into the pricing equation.

Because the AAA is valid at any step of time during the life of an investment, the RAP methodology can formulate the expected return on investment, rate of expected investment costs, and the rate of expected risk premiums at any given time in the future as a function of price,

time, sensitivity, and other parameters.

Suppose that an investor prices an option O on an underlying security S , which follows the typical lognormal process with the drift (or the trend) μ and the volatility σ :

$$(2) \quad dS = \mu S dt + \sigma S d\tilde{z}$$

Here the term dt represents a small increment of time, $d\tilde{z}$ represents a normalized stochastic process, and dS represents a change of the price of the underlying security over this time interval dt . By forming a portfolio P consisting of a unit of the option O and $-\delta$ units of the underlying security S , the portfolio P can be expressed as

$$(3) \quad P = O - \delta S$$

and is subject to the AAA. The amount δ (delta) is usually called the hedge.

In the current RNP models, this portfolio's delta is adjusted continuously and without costs. In the present invention, however, it may be adjusted at discrete steps of time based on a re-hedging strategy of portfolio P . Though transaction costs occur at these discrete steps of time, the present invention spreads the costs over the entire time interval between re-hedging points, similar to the customary practice with the dividend rate, thus defining a rate of transaction costs, r_{TC} . These costs may include, but are not limited to, ask bid spread, slippage, brokerage and exchange fees, short sales taxes, etc. There may be other trading costs associated with the maintenance of the portfolio. For example, short-selling of δ units of stock accrues fees for borrowing the stock in a rate r_B , resulting in a cost $r_B \delta S dt$ over a small time interval dt .

Similarly, a short position in a dividend paying stock or an index with dividend yield r_D results in the costs $r_D \delta S dt$ over a small time interval dt .

The invested amount in this portfolio could have been invested in the government bonds resulting in a riskless return r . For portfolio P consisting of δ units of stock S and option O on this stock, this amount may be equal to $O - \delta S$. For portfolio P consisting of δ units of futures S and an option O on this futures, this amount may be equal to just O . Because the portfolio P cannot be hedged continuously, its value has positive variance, and thus cannot be considered riskless. It is subject to the positive variance risk: the higher the portfolio variance the higher the

risk. The investment should thus generate a risk premium r_{VAR} proportional to this variance risk.

Besides the positive variance risk, the investment is also exposed to risks that the market will not follow our assumptions. A possibility of an option price O jump without a change in the underlying value S generates implied volatility risk. The investment should thus generate a risk premium rate r_{σ} proportional to this implied volatility risk. A possibility of a price gap in the underlying security S generates the gap risk. The investment should therefore generate a risk premium rate r_{GAP} proportional to this gap risk. In the presence of other considered risks, for instance: credit risks, interest rate risks, liquidity risks, daily price limit risks, etc., we would include the corresponding risk premiums. The value of the risk premium is not restricted to the positive values. A negative risk premium corresponds to a negative risk exposure.

In the present invention, the expected return $E[dP]$ on the investment in the portfolio P over a short time interval dt can be expressed as:

$$(4) \quad E[dP] = E[dO - \delta dS]$$

Following the AAA the "Risk-Adjusted Pricing" (RAP) template equation for an option on a stock or an index S is:

$$(5) \quad E[dO - \delta dS] = [r(O - \delta S)dt] + [r_{TC}dt + r_B \delta S dt + r_D \delta S dt] + [r_{VAR}dt + r_{\sigma}dt + r_{GAP}dt],$$

where the rate of borrowing fees r_B and the dividend yield r_D are zero for negative values of δ .

Similarly, the RAP template equation for an option on a futures contract S can be written as:

$$(6) \quad E[dO - \delta dS] = [rOdt] + [r_{TC}dt] + [r_{VAR}dt + r_{\sigma}dt + r_{GAP}dt].$$

From the assumption about the underlying security price process: $dS = \mu S dt + \sigma S d\tilde{z}$ and from the Itô's lemma of stochastic calculus, the stochastic differential equation is:

$$(7) \quad dO = \partial_t O dt + \frac{1}{2} \sigma^2 S^2 \partial_{SS}^2 O dt + \partial_S O dS$$

Re-hedging of the portfolio P over a time interval dt to the value $\delta = \partial_S O$ makes the expected value of $E[dO - \delta dS]$ equal to:

$$(8) \quad E[\partial_t O dt + \frac{1}{2} \sigma^2 S^2 \partial_{SS}^2 O dt + \partial_S O dS - \delta dS] = \partial_t O dt + \frac{1}{2} \sigma^2 S^2 \partial_{SS}^2 O dt,$$

and at the same time it minimizes the variance of the portfolio. Re-hedging consistently to a different delta δ would make the expectation term $E[dO - \delta dS]$ equal to:

$$(9) \quad \partial_t O dt + \frac{1}{2} \sigma^2 S^2 \partial_{SS}^2 O dt + (\partial_S O - \delta) \mu S dt,$$

and it would increase the portfolio variance by $(\partial_S O - \delta)^2 \sigma^2 S^2 dt$. For investors with strong trend expectations, it is thus optimal to re-hedge to a trend-adjusted delta that differs from the value $\partial_S O$.

In general, it is difficult to estimate the future trend term μ from the market. It is a subjective issue. If the trend was known in advance and had a significant value, the underlying price would have been already adjusted accordingly. There is much uncertainty about trends. Each time somebody buys a security, there is somebody who sells it. Similarly, some investors might under-hedge, while others might over-hedge. For the pricing purposes we could assume that the portfolio P is being hedged with the delta given by the value $\partial_S O$.

The rate of the transaction costs r_{TC} related to the re-hedging of the portfolio typically depend on the momentary market conditions that include the size of the ask bid spread, liquidity of the market, and size of the trade. Since large orders tend to move the market, typically the transaction costs per unit contract will be a non-increasing function of the total size of the trade. If the function C denotes an aggregate trading cost per contract as a function of the trading size for the individual market, then the rate of the transaction costs between two re-hedging points over the time interval Δ_t can be estimated as:

$$(10) \quad r_{TC} = \frac{C(N|\Delta\delta|)\Delta\delta}{\Delta_t},$$

where the expected actual size $E[\Delta\delta]$ of the re-hedging trade, i.e. the number of traded units of the security S , is given by the formula:

$$(11) \quad E[|\Delta\delta|] \cong \sqrt{\frac{2}{\pi}} \sigma S |\Gamma| \frac{1}{\sqrt{\Delta_t}},$$

that can be found in the article by Milan Kratka, RISK, April 1998, at 67-71, shown in Figure 7, which is herein incorporated by reference in its entirety. Assuming that the prevailing rate of transaction costs in the market is constant, in the present invention the rate of the transaction cost is equal to:

$$(12) \quad r_{TC} = C \sqrt{\frac{2}{\pi}} \sigma S |\Gamma| \frac{1}{\sqrt{\Delta_t}}.$$

Notably, the longer the time between re-hedging Δ_t , the lower the trading costs rate r_{TC} .

To estimate the rate of the risk premium r_{VAR} compensating the investor for taking the positive variance risk, note that the higher the variance the higher the risk premium. A variance of a normal process increases linearly in time, and the rate of variance risk premium can be modeled by a linear function of the portfolio variance:

$$(13) \quad r_{VAR} \Delta_t = R \cdot \text{var}(\Delta P),$$

where R is the variance risk coefficient, which can also be viewed as a risk aversion coefficient. The higher it is, the more risk premium should be collected for the variance risk. The variance of the portfolio $\text{var}(\Delta P)$ between two re-hedging points with rehedging to $\partial_s O$ over an interval Δ_t can be mathematically estimated as:

$$(14) \quad \text{var}(\Delta P) \cong 3\sigma^4 S^4 \Gamma^2 \Delta_t^2,$$

where $\Gamma = \partial_{ss}^2 O$ as it is standard in the industry. Since the adjustment to the portfolio is done in discrete steps, measure of the variance of the portfolio between adjustment may be subject to a definition discussion. Nevertheless, the dimensional analysis of the expected formula for the variance confirms the proportionality of the variance to $\sigma^4 S^4 \Gamma^2 \Delta_t^2$. Therefore in the present invention the rate of the variance risk premium is:

$$(15) \quad r_{VAR} = 3R\sigma^4 S^4 \Gamma^2 \Delta_t.$$

Notably, the longer the time between re-hedging Δ_t , the higher the variance, and the higher the rate of the variance risk premium r_{VAR} . Since the variance of the hedged portfolio P should be less than a variance of its individual components, an upper bound for the variance risk premium from the variance of the underlying security is:

$$(16) \quad r_{VAR} \leq R\delta^2 \sigma^2 S^2.$$

This estimate is actual typically close to expiration of the option O , when Γ becomes large.

The risk premium related to the implied volatility risk exposure depends on a probability that the implied volatility σ may change, and it depends on the sensitivity of the portfolio value to this volatility change. The sensitivity of the portfolio value P to the change in σ is equal the sensitivity of the option value O to the change in σ . Thus, in the present invention, the implied

volatility risk premium rate as:

$$(17) \quad r_{\sigma} = \nu \partial_{\sigma} O,$$

where ν is the coefficient of dependency.

The risk premium r_{GAP} for the underlying price gap exposure should be proportional to the probability of the price gap, as well as to the size of the price gap. Due to the Itô's lemma, the change of the portfolio value due to the price gap ΔS is equal to:

$$(18) \quad \Delta P = \frac{1}{2} \Gamma (\Delta S)^2,$$

the gap risk premium can be modeled as:

$$(19) \quad r_{GAP} = g \frac{1}{2} \sigma^2 S^2 \Gamma,$$

where the expected price gap size is scaled.

If the financial markets are efficient, then the market pricing will reflect the market value of risk, average prevailing trading costs, as well as competitive re-hedging strategies. There is never a guarantee that the free market will value options in any particular way. But we can assume that it does, build a pricing model and use it to verify our assumptions. The optimal re-hedging strategy in the market dictates minimization of the transaction costs r_{TC} and the premium for the positive portfolio variance r_{VAR} . This leads to a simple algebraic optimization task yielding the solution in terms of optimal re-hedging time interval:

$$(20) \quad \Delta_t = \frac{k^2}{\sigma^2 S^2 \Gamma^{\frac{2}{3}}},$$

where the coefficient $k = \left(C / 3R\sqrt{2\pi} \right)^{\frac{1}{3}}$. This optimal re-hedging time interval corresponds to an optimal re-hedging strategy defined by re-hedging with every change of:

$$(21) \quad \Delta_S = \pm \sigma S \sqrt{\Delta_t} = \pm k \cdot \Gamma^{-\frac{1}{3}}$$

in the underlying security S .

This optimal strategy balances prevailing trading costs r_{TC} against the positive variance risk premium rate r_{VAR} . Taking into account that the variance estimate is bounded from above by the variance of the underlying security component of the portfolio, in the present invention the quantitative enumeration of all individual components in the template RAP equation for the

option on a stock or an index can be summarized as:

$$(22) \quad \Theta + \frac{1}{2} \sigma^2 S^2 \Gamma M = r(O - \delta S) + (r_B + r_D) \delta S + \nu V,$$

where the RAP volatility adjustment coefficient is denoted by:

$$(23) \quad M = 1 + g - \min \left(m \Gamma^{\frac{1}{3}}, 2R \delta^2 \Gamma^{-1} \right)$$

5 the time decay sensitivity theta is denoted by $\Theta = \frac{\partial O}{\partial t}$, the hedging sensitivity delta by $\delta = \frac{\partial O}{\partial S}$, the gamma sensitivity $\Gamma = \frac{\partial^2 O}{\partial S^2}$, and volatility sensitivity by vega by $V = \frac{\partial O}{\partial \sigma}$. The coefficient M

shall always be positive to preserve the phenomenological interpretation of the RAP equation.

Note that the borrowing fees rate r_B is zero when delta is negative. Similarly, in the present invention, the RAP equation for the option on a futures contracts can be written as:

$$(24) \quad \Theta + \frac{1}{2} \sigma^2 S^2 \Gamma M = rO + \nu V.$$

These RAP equations are partial differential equations that can be solved by using various available numerical methods. These RAP equations are typically coupled by so called boundary conditions. These boundary conditions usually arise as a result of the option specifications. Similarly, numerical methods are used for handling extra conditions that arise from the options specifications. For instance when dealing with an early exercise, the numerical solution to the RAP equation will make sure that the value of the option at any particular time does not fall below its exercise value during typical market conditions.

The model coefficients in the RAP equations, such as r , σ , or m , do not need to be constant, but for instance they may be functions of the time t . The RAP methodology includes as a special case the idealistic RNP models. For example, if the possibility of continuous costless hedging without presence of any risks is assumed, the AAA becomes same as the CAA, and the RAP model becomes identical to the Black-Scholes model. On the other hand, the phenomenological explanation and tangible modeling of individual risks and costs included in the RAP methodology allows the RAP models to better explain market pricing even in changing or volatile market environment. The RAP methodology offers not only more precise derivatives

pricing tools, but also offers more advanced risk management tools through its computation of the derivatives sensitivities. The RAP methodology offers a direct way to extract more specific information about value of the market risks and prevailing costs via the inverse problem of fitting the RAP model parameters with sufficient amount of the market data, which can be very
5 useful for the better risk monitoring and portfolio management.

Figure 1 shows a flow chart of the preferred embodiment of RAP evaluation software that computes the value of the derivative by solving the respective RAP equation for that particular derivative. At block 10, derivative characteristics are input for the particular derivative whose value is being calculated. For example, if the derivative is an option, the derivative characteristics may include the type of the option (vanilla puts or calls, spreads, or exotics), the payoff profile, time to expiration, and early exercise feature (American, where the right to exercise the option applies at any point in time until maturity of the option, European, where the right to exercise the option applies only at maturity of the option, or Bermudian, where the right to exercise the option applies only at specific points in time). These characteristics are input manually by the user or automatically by an application software, such as Infinity™ ,
10 PCQuote™, or MicroHedge™.

At block 15, model parameters, like derivative characteristics, are input manually by the user or automatically by application software. In contrast to derivative characteristics, which describe the particular derivative, model parameters describe the behavior of that particular
15 derivative in given markets (equity market, interest rate market, commodity market, foreign exchange market, etc.). For an option, the model parameters could include volatility, risk premium coefficients, trading costs, and risk-free rate.

At block 20, the computation precision is established by setting the resolution in terms of the number of time-steps from the maturity time to the current time, and the number of price-
20 steps and the lower and upper bounds for the underlying price. The lower the resolution, the faster the computation. The higher the resolution, the more precise the computation. These functions described in blocks 10, 15, and 20 can be implemented as an input unit.

Once the derivative characteristics, model parameters, and computation precision are input, the value of the derivative at maturity is initialized by the payoff function of this derivative at block 25. For example, the payoff function of the plain vanilla call option is equal to the maximum value of zero and the difference between the underlying security price at the maturity time and the strike price of the call option. By initializing the value of the derivative at maturity, this necessarily establishes the maturity time at block 30, which is subsequently used in block 35 with the input time-step from block 20 to establish the new time at which the value of the derivative will be computed using the RAP equation for that specific derivative. In block 40, the new time is compared to the current time, and if the new time is greater than or equal to the current time, then the value of the derivative is calculated in block 45 at the new time using the RAP equation for that respective derivative.

After the value of the derivative is calculated, the software returns to block 35 to calculate a new time. That new time is then again compared to the current time in block 40 as explained above. When the new time is less than the current time, the value of the derivative is computed up to the current time and displayed as an output at 50. The functions described in blocks 25, 30, 35, 40, and 45 can be implemented as a processing module and block 50 can be implemented as an output unit.

In block 45, in the preferred embodiment, the RAP equation is solved by the semi-implicit Crank-Nicolson finite difference scheme with the explicit method used for the nonlinear terms in the RAP equation. The RAP equation may be solved also by using other finite difference schemes, finite element, Runge-Kutta ordinary differential equations, or other numerical methods. In block 45, not only are the derivative's values computed, but also its partial derivatives.

Figure 6 shows the source code for a preferred embodiment of the RAP evaluation software, herein incorporated by reference. The software is useful for evaluating and pricing financial derivatives with use of a computer or any other electronic calculation aid and allows entry by a user of various inputs, while producing a tangible price output. Some preferred

outputs of such a device include a discrete price or value of a derivative, graphical representations of price, as shown in Figures 2-5, and reports of values of derivatives. Of course, the output can be modified, added to, or supplemented in myriad ways depending on the user's preferences. Further, as stated above, the method and device of the present invention allows the user to input discrete risk factors and utilize those risk factors in any ways preferred by the user to produce the above-mentioned results.

The graphical representations shown in Figures 2-5 can be generated by using the values calculated for the derivative by the solution of the RAP equation and entering the values into a commercial graphing package, such as Microsoft™ EXCEL™.

With reference to Figure 6, an exemplary computer code listing has been provided commensurate with the disclosed invention. The source code listing of Figure 6 is written in C/C++ and is best utilized with respect to a call or put security based on a European variety option. The code computes the value of a security based on a time reference as shown in the flow chart of Figure 1, above. The computer code can be run on any well known computer system as is well known to those of ordinary skill in the art. The code first requests the following inputs: 1) T, time to maturity in years; 2) K, strike price in dollars; 3) S, current price of the underlying security in dollars; 4) Y, option type (put or call); 5) sig, implied underlying volatility; 6) mu, expected trend rate; 7) R, variance risk premium; 8) C, trading cost/slippage coefficient; 9) v, volatility risk premium coefficient; 10) g, gap risk coefficient; 11) r, risk free rate; 12) rb, borrowing fees rate; 13) rd, dividend yield rate; 14) nt, number of time steps; 15) ns, number of price steps; 16) ls, lower bound of underlying price; and 16) us, upper bound of underlying price. The program initializes the variables, and creates the appropriate array of matrix coefficients. The program then verifies that the coefficients are within their normal ranges. The program then processes the variables and numerical inputs using the RAP method, as stated in the comments in the program itself, as shown. The program then stores the two-dimensional array as computed, and produces a value of the security. The program steps also compute and produce required theoretical values of securities in real time, and display or report

on the same. Further, use of the computer program model and steps allow real-time reporting and monitoring of the value of the underlying risks, to extract the value of the risk embedded in the security, to generate and display a delta hedging parameter (sensitivity of the option value to change in the value of the underlying security), as well as other potential numbers or reports.

5 Further, use of the direct outputs can be made utilizing standard software tools, like Microsoft™ EXCEL™, to produce graphical representations of risk and value utilizing other models. Some examples of such are shown in Figures 2 through 5. These graphs were produced by inputting the values derived with the RAP methodology into Excel, with the desired values for the axes as shown, and running the graph routines having the shown data. As can be imagined, the reports, graphs, and other results are limited only by the requirements of the user. While particular elements, embodiments and applications of the present invention have been shown and described, it is understood that the invention is not limited thereto since modifications may be made by those skilled in the art, particularly in light of the foregoing teaching. It is therefore contemplated by the appended claims to cover such modifications and incorporate those features that come within the spirit and scope of the invention.